PERGAMON

International Journal of Solids and Structures 36 (1999) 3093–3100

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

Author's closure

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Received 5 February 1998

The author wishes to thank Messrs Borino and Polizzotto for their interest in his paper on the nonlocal softening bar, and in particular for their critical comments, which have a bearing on the general applicability of the maximum dissipation principle, and specifically on the validity of the principle in general nonlocal plasticity. In the following, various issues of disagreement between the author and Borino and Polizzotto are discussed.

1. The nonlocal plastic strain

Although for reasons of simplicity, ε is identified in the paper as infinitesimal strain, it should be noted that the decomposition of strain into elastic and plastic parts is not performed. In infinitesimal plasticity plastic strain is a defined quantity (due to the decomposition rule), whereas in finite plasticity no general agreement on how plastic strain should be defined exists. In general nonlocal plasticity it would seem unlikely indeed, unless deformation is homogeneous, that a prescription unambiguously identifying plastic strain can be found. Nevertheless, Borino and Polizzotto assert that, in theories of nonlocal plasticity suitable for application to localized failure, plastic strain should be treated as local. They refer to results reported in the papers of Bažant and Pijaudier-Cabot (1988) and of Bažant and Lin (1988). These papers are not of much help in this respect, however, since the Bažant and Pijaudier-Cabot paper concerns nonlocal elastic damage (with local strain), and Bažant and Lin in fact treat plastic strain as nonlocal (while keeping total strain local).

2. The dissipation function

Borino and Polizzotto remark that the author presents (26) and (27) without introducing any nonlocality residual. However, (26) is merely a definition and for (27) it is made clear that this relationship applies to a localized form of the dissipation inequality having a vanishing nonlocality residual. One should note, in this context, that (10) is not a general statement either but represents a special form of the second law of a nonlocal nonpolar mechanical theory, a form pertaining to a locally mass closed body for which the nonlocality residuals for both linear and rotational momentum vanish.

PII: S0020–7683(98)00076–6

Whereas the author treats (27) as only a sufficient condition for the second law to hold true, Borino and Polizzotto claim that it is also a necessary one. They appear to also disagree with the author concerning the interpretation of nonlocality residuals (as will subsequently be discussed). The author would first note, however, that the argument Borino and Polizzotto use as a basis for concluding that the nonlocality residual P vanishes identically is questionable. They assert that the expression for the thermodynamic forces $\mathbf{S}$ and Q, as given by (98), follows from (97). Certainly, (98) does contain conditions sufficient for (97) to be fulfilled. However, they are not necessary since $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$ cannot be treated as completely independent variables. In view of (101a), one may assume that

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\hat{\boldsymbol{p}}(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h), \quad \dot{\kappa} = \dot{\lambda}\hat{h}(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h), \tag{114}$$

with $\dot{\lambda} > 0$ at plastic loading. Hence, the condition (97) requires (plastic loading presupposed) that

$$\int_B [(\mathbf{S} - \overline{\{\boldsymbol{\sigma}\}}_p) \cdot \hat{\boldsymbol{p}} + (Q - \overline{\{q\}}_h)\hat{h}]\,\mathrm{d}V = 0, \tag{115}$$

from which (98) does not necessarily follow.

If P does not vanish identically, (26) subject to condition (27) is not in agreement with (93). The following argument might help to clarify the author's interpretation of nonlocality residuals in nonlocal continuum mechanics. Assume $(24)_1$ as the plastic dissipation and define a dissipation density as in (26) (bilinear in $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$). Localization of the global dissipation inequality $\mathbb{D}^p \geqslant 0$ then leads to

$$D^p + P \geqslant 0, \quad \int_B P\,\mathrm{d}V = 0, \tag{116}$$

or in equivalent *local* forms

$$\int_{\mathscr{P}} (D^p + P) \geqslant 0, \quad \int_B P\,\mathrm{d}V = 0, \tag{117}$$

and

$$\int_{\mathscr{P}} D^p\,\mathrm{d}V - \int_{B-\mathscr{P}} P\,\mathrm{d}V \geqslant 0, \tag{118}$$

where $\mathscr{P}$ is an arbitrary part of B. In view of (118)—where D^p represents the rate of energy dissipated locally at $\mathbf{x}$—the residual P may be interpreted as the rate at which energy is supplied to a particle at $\mathbf{x}$ due to the presence of the rest of the body. Seen in this light, (26) is an appropriate definition of the plastic dissipation density, whereas (27) is the localized form of $(24)_1$ corresponding to a vanishing nonlocality residual. For instance, choosing

$$P = \boldsymbol{\mu\sigma} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle - \overline{\{\boldsymbol{\mu\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + vq\langle \dot{\kappa} \rangle - \overline{\{vq\}}_h \dot{\kappa}, \tag{119}$$

where $\boldsymbol{\mu}(\mathbf{x})$ and $v(\mathbf{x})$ are arbitrary tensorial and scalar functions, the dissipation inequality becomes

$$\overline{\{(\mathbf{1} - \boldsymbol{\mu})\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{(1 - v)q\}}_h \dot{\kappa} + \boldsymbol{\mu\sigma} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle + vq\langle \dot{\kappa} \rangle \geqslant 0, \tag{120}$$

which is only identical with (27) if restricted to local theory. Choosing $\boldsymbol{\mu}(\mathbf{x}) = \mathbf{1}$ and $v(\mathbf{x}) = 1$ yields the inequality

$$\boldsymbol{\sigma} \cdot \langle \dot{\boldsymbol{\varepsilon}}^p \rangle + q \langle \dot{\kappa} \rangle \geqslant 0, \tag{121}$$

where the left-hand side is identified as the integrand of $(24)_2$ (but of course not equal to D^p). Apparently, the global inequality $\mathbb{D}^p \geqslant 0$ results in an arbitrary number of local statements, one of which is (27), corresponding to $P = 0$.

3. The plastic flow laws

Recall that in the case of restricted nonlocality the independent variables are $\boldsymbol{\varepsilon}$, $\langle \boldsymbol{\varepsilon}^p \rangle$ and $\langle \kappa \rangle$. In addition to the constitutive assumption (19), a yield function must be introduced and constitutive equations for the rates of plastic strain and strain hardening (flow rules or flow laws) be postulated. In the author's view, the corresponding yield criterion (in strain space) can be formulated as

$$g(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle) \leqslant 0, \tag{122}$$

whereas the flow rules along a strain trajectory can be assumed to have the form

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda} \boldsymbol{p}(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle), \quad \dot{\kappa} = \dot{\lambda} h(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle). \tag{123}$$

Here $\dot{\lambda} = \dot{\lambda}\,(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle; \dot{\boldsymbol{\varepsilon}})$ is a plastic multiplier, one which may be assumed to depend linearly on $\dot{\boldsymbol{\varepsilon}}$, and which at plastic loading is positive. What is the 'normality rule' in this general case? Obviously, the procedure suggested by Borino and Polizzotto is not applicable, the pertinent thermodynamic forces being displayed in $(16)_1$. In a corresponding local theory, a normality rule can be derived by employing a work assumption of one form or another (see e.g. Casey and Naghdi, 1984) or by invoking the principle of maximum dissipation. To arrive at the normality rule in its classical form, however, it is necessary to look at a restricted class of elastic–plastic materials. Assume that (21) is valid (invertibility) and define a yield function in stress space by the relation

$$g(\boldsymbol{\varepsilon}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle) = g(\tilde{\boldsymbol{\varepsilon}}(\boldsymbol{\sigma}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle), \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle) = f(\boldsymbol{\sigma}, \langle \boldsymbol{\varepsilon}^p \rangle, \langle \kappa \rangle). \tag{124}$$

For the special class of materials that comply with (22) and (23), the corresponding yield criterion (in stress space) becomes

$$f(\boldsymbol{\sigma}, q) \leqslant 0, \tag{125}$$

where for convenience $\langle \kappa \rangle$ has been replaced by q [defined by (25)]. In local theory the normality rule is of the form $\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\, \partial f / \partial \boldsymbol{\sigma}$ corresponding to (101a) (the flow rule for plastic strain in associated plasticity). Further, still assuming local theory, (24) and (26) may be replaced by the single equation

$$D^p = \boldsymbol{\sigma} \cdot \dot{\boldsymbol{\varepsilon}}^p + q \dot{\kappa}, \tag{126}$$

whereas (125) appears in unaltered form ($\boldsymbol{\sigma}$ and $q = -H\kappa$ now being local thermodynamic forces). By invoking the maximum dissipation principle, it can be shown that the flow rules become

$$\dot{\boldsymbol{\varepsilon}}^p = \dot{\lambda}\frac{\partial f}{\partial \boldsymbol{\sigma}}, \quad \dot{\kappa} = \dot{\lambda}\frac{\partial f}{\partial q}, \tag{127}$$

subjected to the Kuhn–Tucker conditions

$$f(\boldsymbol{\sigma}, q) \leqslant 0, \quad \dot{\lambda} \geqslant 0, \quad \dot{\lambda} f(\boldsymbol{\sigma}, q) = 0. \tag{128}$$

Here the thermodynamic forces related to $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$ are $\boldsymbol{\sigma}$ and q, respectively. Adopting the terminology Borino and Polizzotto employ, (127) together with (128) may be regarded then as a generalized normality rule in local plasticity.

For the corresponding nonlocal case, Borino and Polizzotto assert that (101a) and (101b) follow by a simple application of this normality rule. However, such an argument is questionable. Going on then to nonlocality, one can note that the set of admissible states is given by (28), where $\boldsymbol{\sigma}$ and q are not (as Borino and Polizzotto claim) 'local stresses' but are nonlocal quantities given by (23) and (25), and that (126) can be replaced by (26) (corresponding to a zero-residual of nonlocality). Apparently, (26) and (28) do not appear in a format corresponding to (125) and (126) in the local case. Hence, in the author's view, the 'normality rule' of local plasticity cannot be applied. However, as Borino and Polizzotto also note, (101a) and (101b) can be derived by invoking the maximum dissipation principle in the form displayed by (109), i.e. after a change of independent variables. See next section for a further discussion of this.

4. The maximum dissipation principle

The principle (or postulate) of maximum dissipation in classical plasticity is a highly restrictive assumption, allowing the associated flow rule (the normality rule) for plastic strain to be derived. By invoking a generalized form of the classical principle (127) and (128) can be derived as well. Is it possible then to also apply the principle in nonlocal plasticity so as to derive associated flow rules that comply with (24), (28) and (123) (or more generally with (14), (15), (122) and (123))? If the principle is stated as

$$\max \mathbb{D}^p[\boldsymbol{\sigma}, q; \langle\dot{\boldsymbol{\varepsilon}}^p\rangle, \langle\dot{\kappa}\rangle], \quad f(\boldsymbol{\sigma}, q) \leqslant 0 \quad \text{in } B, \tag{129}$$

where

$$\mathbb{D}^p = \int_B [\boldsymbol{\sigma} \cdot \langle\dot{\boldsymbol{\varepsilon}}^p\rangle + q\langle\dot{\kappa}\rangle]\,\mathrm{d}V, \tag{130}$$

it follows by the Lagrange multiplier method that the flow rules can be written in the form given by (108a), subject to the conditions in (108b). This is a trivial result of local theory, the extension to nonlocal plasticity in effect being empty. Since $\dot{\boldsymbol{\varepsilon}}^p$ and $\dot{\kappa}$ only appear here disguised as $\langle\dot{\boldsymbol{\varepsilon}}^p\rangle$ and $\langle\dot{\kappa}\rangle$, it is merely a question of renaming variables. An appropriate Lagrangian can be written as

$$L^p(\boldsymbol{\sigma}, q, \dot{\lambda}) = -\boldsymbol{\sigma} \cdot \langle\dot{\boldsymbol{\varepsilon}}^p\rangle - q\langle\dot{\kappa}\rangle + \dot{\lambda} f(\boldsymbol{\sigma}, q), \tag{131}$$

and (129) can be solved by optimality conditions corresponding to (31), which implies (108a) and (108b). Hence, the application of the principle in this form does not lead to flow rules that comply with (123). In the original paper, the author failed to see that replacing $(24)_2$ by $(24)_1$ has no influence whatever on the result of the optimality problem. For the latter, the principle can be formulated as

$$\max \mathbb{D}^p[\boldsymbol{\sigma}, q\,; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}], \quad f(\boldsymbol{\sigma}, q) \leqslant 0 \quad \text{in } B, \tag{132}$$

where now

$$\mathbb{D}^p = \int_B [\overline{\{\boldsymbol{\sigma}\}}_p \cdot \dot{\boldsymbol{\varepsilon}}^p + \overline{\{q\}}_h \dot{\kappa}]\, \mathrm{d}V. \tag{133}$$

The corresponding Lagrangian is given by

$$\begin{aligned}\mathbb{L}^p(\boldsymbol{\sigma}, q, \dot{\lambda}) &= -\mathbb{D}^p(\overline{\{\boldsymbol{\sigma}\}}_p, \overline{\{q\}}_h\,; \dot{\boldsymbol{\varepsilon}}^p, \dot{\kappa}) + \int_B \dot{\lambda} f(\boldsymbol{\sigma}, q)\, \mathrm{d}V \\ &= \int_B \left[-\int_B \tilde{w}^p(\mathbf{x}, \mathbf{z})\boldsymbol{\sigma}(\mathbf{z}) \cdot \dot{\boldsymbol{\varepsilon}}^p + \tilde{w}^h(\mathbf{x}, \mathbf{z}) q(\mathbf{z}) \dot{\kappa}\, \mathrm{d}V(\mathbf{z}) + \dot{\lambda} f(\boldsymbol{\sigma}, q) \right] \mathrm{d}V(\mathbf{x})\end{aligned} \tag{134}$$

and its variation by

$$\begin{aligned}\delta \mathbb{L}^p = &\int_B \left[-\int_B \tilde{w}^p(\mathbf{x}, \mathbf{z}) \delta\boldsymbol{\sigma}(\mathbf{z}) \cdot \dot{\boldsymbol{\varepsilon}}^p\, \mathrm{d}V(\mathbf{z}) + \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \delta\boldsymbol{\sigma} \right] \mathrm{d}V(\mathbf{x}) \\ &+ \int_B \left[-\int_B \tilde{w}^h(\mathbf{x}, \mathbf{z}) \delta q(\mathbf{z}) \dot{\kappa}\, \mathrm{d}V(\mathbf{z}) + \dot{\lambda} \frac{\partial f}{\partial q} \delta q \right] \mathrm{d}V(\mathbf{x}) + \int_B \delta\dot{\lambda} f(\boldsymbol{\sigma}, q)\, \mathrm{d}V.\end{aligned} \tag{135}$$

Interchanging **x** and **z** and reversing the order of integration allows (135) to be written as

$$\begin{aligned}\delta \mathbb{L}^p = &\int_B \left[\left(-\int_B (\tilde{w}^p(\mathbf{z}, \mathbf{x}) \dot{\boldsymbol{\varepsilon}}^p(\mathbf{z})\, \mathrm{d}V(\mathbf{z}) + \dot{\lambda} \frac{\partial f}{\partial \boldsymbol{\sigma}} \right) \cdot \delta\boldsymbol{\sigma} \right] \mathrm{d}V(\mathbf{x}) \\ &+ \int_B \left[\left(-\int_B \tilde{w}^h(\mathbf{z}, \mathbf{x}) \dot{\kappa}(\mathbf{z})\, \mathrm{d}V(\mathbf{z}) + \dot{\lambda} \frac{\partial f}{\partial q} \right) \delta q \right] \mathrm{d}V(\mathbf{x}) + \int_B \delta\dot{\lambda} f(\boldsymbol{\sigma}, q)\, \mathrm{d}V.\end{aligned} \tag{136}$$

Hence, in view of (8), the optimality conditions are given by (108a) and (108b). This result is not surprising, since the step taken to pass from (135) to (136) is the same as that which transforms $(24)_1$ into $(24)_2$. As a consequence, unless the set of independent variables is replaced by a set of corresponding conjugated forces [e.g. as those appearing in (109)], it appears to be impossible to apply the maximum dissipation principle (in its classical form) to obtain a non-trivial solution. Although (32) and (33) do not follow from a maximum dissipation principle [extended from local theory and stated as in (129)], one can note that, as long as (15) is satisfied, (32) is consistent with both (125) and general thermodynamics. Since this is the only restriction[1] placed on the constitutive functions of the theory ($\psi, \boldsymbol{\sigma}$ and f), it is difficult to see what Borino and Polizzotto have in mind

[1] Note that convexity of the yield surface is not a sufficient condition for the inequality to hold true for all f.

with regard to the statement that precedes (113). (It is not appropriate to take the left-hand side of (113) as a definition of 'plastic work' in the present theory.)

5. The bar solution

The features of the solution to a nonlocal structural problem are highly affected by the specific choice of attenuation functions due to their influence on the constitutive equations of the material. For the bar solution, the width of the localized zone is essentially determined by the characteristic length l (considered as a material parameter of a nonlocal body), the elastic modulus E and the softening modulus H. Since the structural size effect is apparently a nonlocal property of a body, it is not surprising that the width of the localized zone depends on the length of the bar. This dependence is negligible, however, if the parameter l is small compared with the length of the bar. It should be emphasized that the criterion used to determine the bandwidth is specific to the bar problem and cannot unequivocally be generalized to two- or three-dimensional problems.

In the paper, the attenuation functions are chosen as being rapidly decaying but also as being non-zero everywhere along the bar. Although the latter property is unnecessary, of course, it is suitable for an analytical treatment of the problem. In a numerical solution strategy it would be more convenient to define attenuation functions as only being non-zero within some prescribed distance from a given point (as was done by Bažant and Lin, 1988). Borino and Polizzotto refer again to the paper of Bažant and Lin and its solution of the bar problem and claim that paper to indicate the bandwidth to be a strict material parameter. That problem is not addressed, however, in the Bažant and Lin paper, which deals with numerical solution of two-dimensional nonlocal softening problems and also reports that the bandwidth (in the case of a tensioned rectangular panel) is not constant but varies with loading.

It should be emphasized that the introduction of nonlocality for plastic strain is crucial to the theory dealt with in the paper. Treating plastic strain as local would force the localized zone into a region of vanishing size.

It is interesting to note that all essential features of the bar solution as given in Section 3 and 4 remain unchanged if (32) and (33) are replaced by (101a) and (101b), respectively. If σ_y is uniform along the bar (strain softening may be assumed to be initiated through some imperfection at x_0), the yield function corresponding to (41) becomes

$$f = \overline{\{\sigma\}}_p - \overline{\{\sigma_y\}}_p + \overline{\{q\}}_h = \beta^p(\sigma - \sigma_y) + \int_B \tilde{w}^h(x, z) q(z)\, \mathrm{d}z, \quad \sigma > 0, \tag{137}$$

where (79) has also been used. The flow rules corresponding to (45) may be written as

$$\dot{\varepsilon}^p(x) = \dot{\kappa}(x) = \dot{\lambda}(x) = \dot{B}(x)\delta(x - x_0), \tag{138}$$

and hence

$$\langle \dot{\varepsilon}^p \rangle(x) = \dot{B}(x_0)\tilde{w}^p(x_0, x), \quad \langle \dot{\kappa} \rangle(x) = \dot{B}(x_0)\tilde{w}^h(x_0, x). \tag{139}$$

The corresponding consistency condition leads to the following expression for $\dot{B}$ [to replace (49)] :

$$\dot{B}(x_0) = \frac{\beta^p(x_0)}{\beta^*(x_0)}\frac{\dot{\sigma}}{H}, \tag{140}$$

where

$$\beta^*(x_0) = \int_B (\tilde{w}^h(x_0, z))^2 \, \mathrm{d}z. \tag{141}$$

Hence, in view of (40), $(139)_1$ and (140), it follows that (50) should be replaced by

$$\dot{\varepsilon}(x) = \dot{\sigma}\left[\frac{1}{E} + \frac{1}{H}\frac{\beta^p(x_0)}{\beta^*(x_0)}\tilde{w}^p(x_0, x)\right]. \tag{142}$$

It is easy to show then that the total dissipation given by (67) should be replaced by

$$\mathbb{D}^p = \frac{(\beta^p(x_0))^2}{\beta^*(x_0)}\frac{\dot{\sigma}}{H}\sigma_y, \tag{143}$$

preserving all the characteristic properties called attention to in the paper. It should also be noted that (73) remains valid if the characteristic length l^*_{ch} is defined as

$$l^*_{ch} = \frac{(\beta^p(x_0))^2}{\beta^*(x_0)}. \tag{144}$$

Finally, it can be shown for a wide class of attenuation functions that (138) represents the only nontrivial equilibrium solution of the bar problem.

6. Conclusions

A number of questions have been raised with regard to the principle of maximum dissipation and its role in nonlocal plasticity. The principle in itself represents a highly restrictive assumption which in classical plasticity leads to the flow rules of associated plasticity. In the author's view it is not clear that the principle can be extended to embrace nonlocality in such a way that appropriate flow rules of nonlocal associated plasticity can be derived. In other words, if the principle of maximum dissipation cannot be stated in a form leading to flow rules consistent with (123), doubt can be cast upon its validity in nonlocal plasticity.

The bar solution presented in the paper has the following characteristic features:

- the solution is unique in the sense that strain softening represented by the Dirac distribution (81) appears (for a certain class of attenuation functions) to be the only nontrivial equilibrium solution of the elastic–plastic equations of the problem
- plastic strain needs to be treated as a nonlocal quantity in order to ensure that the localized zone does not shrink into a region of vanishing size
- the width of the localized zone is essentially a material property.

Caution should be observed in any attempt to extend the results of the bar solution to general softening plasticity. Plastic strain, although necessary in the case of the softening bar, cannot be claimed unconditionally to be nonlocal in theories capable of treating localized failure. For instance, for coupled elastic–plastic damage it would appear very likely that the damage variable can be chosen as the only nonlocal quantity present in the model. For general elastic–plastic models (without damage), however, the author is unable to see any obvious reason for considering plastic strain as local.

It should be emphasized, finally, that models based on gradient plasticity and nonlocal plasticity, respectively, cannot be expected to provide identical results, due to the nonequivalence of these theories (as was discussed in the original paper).

References

Casey, J., Naghdi, P.M., 1984. Further constitutive results in finite plasticity. Q. J. Mech. Math. 37(2), 231–259.

Errata

Also add the following corrections to the original paper:

Page 4403, paragraph 1, line 1: Read "Note that if one takes $\tilde{w} = \delta$ (the Dirac delta function) in (8), it follows that the original state functions are recovered".
Page 4413, paragraph 1, line 5: Read "... in which $\tilde{w}^p = \delta \neq \tilde{w}^h$ (and hence $\beta^p(x) = 1$ ...".
Page 4414, paragraph 1, line 2: Read "... when $\tilde{w}^p = \delta$".